

A NEW PROOF OF THE WEAK STRUCTURE THEOREM

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ABSTRACT

We give an elementary and self-contained proof, and a numerical improvement, of a weaker form of the excluded clique minor theorem of Robertson and Seymour, the following. Let $t, r \geq 1$ be integers, and let $R = 49152t^{24}(12t^2 + r)$. An r -wall is obtained from a $2r \times r$ -grid by deleting every odd vertical edge in every odd row and every even vertical edge in every even row. Let G be a graph with no K_t minor, and let W be an R -wall in G . We prove that there exist a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r -subwall W' of W such that $V(W') \cap A = \emptyset$ and W' is a flat wall in $G - A$ in the following sense. There exists a separation (X, Y) of $G - A$ such that $X \cap Y$ is a subset of the vertex set of the cycle C' that bounds the outer face of W' , $V(W') \subseteq Y$, and the graph $G[Y]$ can almost be drawn in the unit disk with the vertices $X \cap Y$ drawn on the boundary of the disk in the order determined by C' . Here almost means that the assertion holds after repeatedly removing parts of the graph separated from $X \cap Y$ by a cutset Z of size at most three, and adding all edges with both ends in Z . Our proof gives rise to an algorithm that runs in polynomial time even when r and t are part of the input instance.

¹Partially supported by NSF under Grant No. DMS-0701077.

²Partially supported by the European Research Council under the European Unions Seventh Framework Programme (FP7/2007-2013)/ERC Grant Agreement no. 279558.

12 June 2012, revised 29 July 2012.

1 Introduction

All graphs in this paper are finite, and may have loops and parallel edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. An *H minor* is a minor isomorphic to H . There is an ever-growing collection of so-called excluded minor theorems in graph theory. These are theorems which assert that every graph with no minor isomorphic to a given graph or a set of graphs has a certain structure. The best known such theorem is perhaps Wagner's reformulation of Kuratowski's theorem [20], which says that a graph has no K_5 or $K_{3,3}$ minor if and only if it is planar. One can also characterize graphs that exclude only one of those minors. To state such a characterization for excluded K_5 we need the following definition. Let H_1 and H_2 be graphs, and let J_1 and J_2 be complete subgraphs of H_1 and H_2 , respectively, with the same number of vertices. Let G be obtained from the disjoint union of $H_1 - E(J_1)$ and $H_2 - E(J_2)$ by choosing a bijection between $V(J_1)$ and $V(J_2)$ and identifying the corresponding pairs of vertices. We say that G is a *clique-sum* of H_1 and H_2 . Since we allow parallel edges, the set that results from the identification of $V(J_1)$ and $V(J_2)$ may include edges of the clique-sum. For instance, the graph obtained from K_4 by deleting an edge can be expressed as a clique-sum of two smaller graphs, where one is a triangle and the other is a triangle with a parallel edge added. By V_8 we mean the graph obtained from a cycle of length eight by adding an edge joining every pair of vertices at distance four in the cycle. The characterization of graphs with no K_5 minor, due to Wagner [19], reads as follows.

Theorem 1.1 *A graph has no K_5 minor if and only if it can be obtained by repeated clique-sums, starting from planar graphs and V_8 .*

There are many other similar theorems; a survey can be found in [1]. Theorem 1.1 is very elegant, but attempts at extending it run into difficulties. For instance, no characterization is known for graphs with no K_6 minor, and there is evidence suggesting that such a characterization would be fairly complicated. Even if a characterization of graphs with no K_6 is found, there is no hope in finding one for excluding K_t for larger values of t .

Thus when excluding an H minor for a general graph H we need to settle for a less ambitious goal—a theorem that gives a necessary condition for excluding an H minor, but not necessarily a sufficient one. However, for such a theorem to be meaningful, the structure it describes must be sufficient to exclude some other, possibly larger graph H' . For planar graphs H this has been done by Robertson and Seymour [9]. To state their theorem we need to recall that the *tree-width* of a graph G is the least integer k such that G can be obtained by repeated clique-sums, starting from graphs on at most $k + 1$ vertices.

Theorem 1.2 *For every planar graph H there exists an integer k such that every graph with no H minor has tree-width at most k . If H is not planar, then no such integer exists.*

This is a very satisfying theorem, because it is best possible in at least two respects. Not only is there no such integer when H is not planar, but no graph of tree-width k has a minor isomorphic to the $(k + 1) \times (k + 1)$ -grid.

But how about excluding a non-planar graph? Robertson and Seymour have an answer to that question as well, but in order to motivate it we need to digress a bit.

1.1 The Two Disjoint Paths Problem

Let G be a graph, and let $s_1, s_2, t_1, t_2 \in V(G)$. The TWO DISJOINT PATHS PROBLEM asks whether there exist two disjoint paths P_1, P_2 in G such that P_i has ends s_i and t_i . There is a beautiful characterization of the feasible instances, which we now describe. First of all, let us assume that G has a cycle C with vertex-set $\{s_1, s_2, t_1, t_2\}$ in order. This we can assume, because the edges of C can be added without changing the feasibility status of the problem. Now if G can be drawn in the plane with C bounding a face, then the problem is infeasible. (Proof. Add a new vertex in the face bounded by C and join it by an edge to every vertex of C . The new graph is planar, and yet if the paths existed, they would give rise to a K_5 minor in G .) So this gives one class of obstructions, but there is another one. A *separation* in a graph G is a pair (A, B) such that $A \cup B = V(G)$, and there is no edge of G with one in $A - B$ and the other in $B - A$. The order of the separation (A, B) is $|A \cap B|$. Now if there exists a separation (A, B) of G of order at most three with $V(C) \subseteq A$, then the vertices in $B - A$ are not very useful. We may choose (A, B) so that some component of $G[B - A]$ includes a neighbor of every vertex in $A \cap B$. In that case the feasibility of the problem remains unchanged if we delete $B - A$ and instead add an edge joining every pair of vertices in $A \cap B$. Let us turn this observation into a definition, but first let us recall that a collection of paths \mathcal{P} are *internally disjoint* if every vertex that belongs to two distinct members of \mathcal{P} is an end of both.

internally
disjoint

Definition Let G be a graph, and let $X \subseteq V(G)$. Let (A, B) be a separation of G of order at most three with $X \subseteq A$ and such that there exist $|A \cap B|$ internally disjoint paths from some vertex of $B - A$ to X . Let H be the graph obtained from $G[A]$ by adding an edge joining every pair of vertices in $A \cap B$. We say that H is an *elementary X -reduction* of G , and we say that it is an *elementary X -reduction determined by (A, B)* . We say that a graph J is an *X -reduction* of G if it can be obtained from G by a series of elementary X -reductions. If C is a subgraph of G , then by an (elementary) C -reduction we mean an (elementary) $V(C)$ -reduction.

elementary
 X -
reduction
 X -
reduction

Thus taking C -reductions does not change the feasibility of the TWO DISJOINT PATHS PROBLEM, and as we are about to see, when no C -reduction is possible, the only obstruction to the existence of the required paths is topological, namely that G can be drawn in the plane with C bounding a face. To state the theorem in a slightly more general form, let us say that a *C -cross* in a graph G is a pair of disjoint paths P_1, P_2 with ends s_1, t_1 and s_2, t_2 , respectively, such that s_1, s_2, t_1, t_2 occur on C in the order listed, and the paths are otherwise disjoint from C . The first version of the promised theorem, obtained in various forms by Jung [4], Robertson and Seymour [10], Seymour [15], Shiloach [16], and Thomassen [18] reads as follows.

C -cross

Theorem 1.3 *Let G be a graph, and let C be a cycle in G . Then G has no C -cross if and only if some C -reduction of G can be drawn in the plane with C bounding a face.*

For applications it is desirable to have a representation of the entire graph G as opposed to some unspecified C -reduction. Formalizing this idea is the subject to the next definition.

Definition Let G be a graph, and let C be a cycle in G . We say that G is C -flat if there exist subgraphs G_0, G_1, \dots, G_k of G , and a plane graph Γ such that for all distinct indices $i, j = 1, 2, \dots, k$ C -flat

- (1) $G = G_0 \cup G_1 \cup \dots \cup G_k$, and the graphs G_0, G_1, \dots, G_k are pairwise edge-disjoint,
- (2) C is a subgraph of G_0 and G_0 is a subgraph of Γ with the same vertex-set as Γ ,
- (3) the cycle C bounds the outer face of the plane graph Γ ,
- (4) $|V(G_i) \cap V(G_0)| \leq 3$; if $V(G_i) \cap V(G_0) = \{u, v\}$, then u and v are adjacent in Γ , and if $V(G_i) \cap V(G_0) = \{u, v, w\}$, then some finite face of Γ is incident with u, v, w and no other vertex, and
- (5) $V(G_i) \cap V(G_j) \subseteq V(G_0)$.

Using the above definition we can extend Theorem 1.3 as follows.

Theorem 1.4 *Let G be a graph, and let C be a cycle in G . Then the following conditions are equivalent:*

- (1) G has no C -cross,
- (2) some C -reduction of G can be drawn in the plane with C bounding a face, and
- (3) G is C -flat.

The equivalence of (2) and (3) is not hard to see, but we omit the details, because we do not need it in this paper.

1.2 The Weak Structure Theorem

We are now ready to formulate the weaker version of the excluded K_t theorem of Robertson and Seymour [11, Theorem 9.8]. Let us begin by describing it informally. We use $[r]$ to denote $\{1, 2, \dots, r\}$. Let $r \geq 2$ be an integer. An $r \times r$ -grid is the graph with vertex-set $[r] \times [r]$ in which (i, j) is adjacent to (i', j') if and only if $|i - i'| + |j - j'| = 1$. An *elementary r -wall* is obtained from the $2r \times r$ -grid by deleting all edges with ends $(2i - 1, 2j - 1)$ and $(2i - 1, 2j)$ for all $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, \lfloor r/2 \rfloor$ and all edges with ends $(2i, 2j)$ and $(2i, 2j + 1)$ for all $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, \lfloor (r - 1)/2 \rfloor$ and then deleting the two resulting vertices of degree one. An *r -wall* is any graph obtained from an elementary r -wall by subdividing edges. In other words each edge of the elementary r -wall is replaced by a path; those replacement paths will be called *segments*. Figure 1 shows an elementary 4-wall. Walls are harder to describe, but they are easier to work with, because if a graph has a $2r \times 2r$ -grid minor, then

elementary
 r -wall

r -wall

segment

it has a subgraph isomorphic to an r -wall. Let W be an r -wall, where W is the subdivision of an elementary wall Z . Let X be the set of vertices of W that correspond to vertices (i, j) of Z with $j = 1$, and let Y be the set of vertices of W that correspond to vertices (i, j) of Z with $j = r$. There is a unique set of r disjoint paths Q_1, Q_2, \dots, Q_r in W , such that each has one end in X and one end in Y , and no other vertex in $X \cup Y$. We may assume that the paths are numbered so that the first coordinates of their vertices are increasing. We say that Q_1, Q_2, \dots, Q_r are the *vertical paths* of W . There is a unique set of r disjoint paths with one end in Q_1 , the other end in Q_r , and otherwise disjoint from $Q_1 \cup Q_r$. Those will be called the *horizontal paths* of W . Let P_1, P_2, \dots, P_r be the horizontal paths numbered in the order of increasing second coordinates. Then $P_1 \cup Q_1 \cup P_r \cup Q_r$ is a cycle, and we will call it the *outer cycle* of W . If W is drawn as a plane graph in the obvious way, then this is indeed the cycle bounding the outer face. The sets $V(P_1 \cup Q_1)$, $V(P_1 \cup Q_r)$, $V(P_r \cup Q_1)$, and $V(P_r \cup Q_r)$ each include exactly one vertex of W ; those vertices will be called the *corners* of W . In Figure 1 the four corners are circled. Finally let W, W' be walls such that W' is a subgraph of W . We say that W' is a *subwall* of W if every horizontal path of W' is a subpath of a horizontal path of W , and every vertical path of W' is a subpath of a vertical path of W .

vertical
paths
horizontal
paths
outercycle
corners
subwall

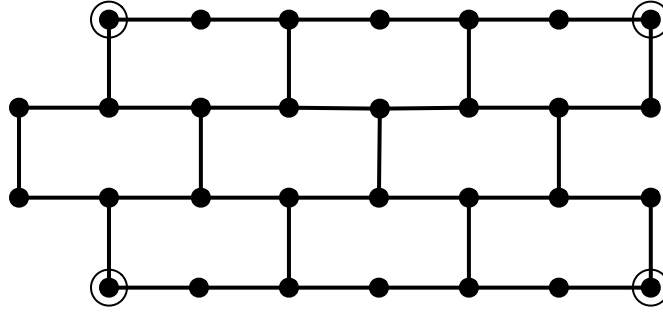


Figure 1: An elementary 4-wall.

Now let W be a large wall in a graph G with no K_t minor. The Weak Structure Theorem asserts that there exist a set of vertices $A \subseteq V(G)$ of bounded size and a reasonably big subwall W' of W that is disjoint from A and has the following property. Let C' be the outer cycle of W' . The property we want is that C' separates the graph $G - A$ into two graphs, and the one containing W' , say H , can be drawn in the plane with C' bounding a face. However, as the discussion of the previous subsection attempted to explain, the latter condition is too strong. The most we can hope for is for the graph H to be C' -flat. That is, in spirit, what the theorem will guarantee, except that we cannot guarantee that all of C' be part of a planar C' -reduction of H . The correct compromise is that some subset of $V(C')$ separates off the wall W' , and it is that subset that is required to be incident with one face of the planar drawing. Here is the formal definition.

Definition Let G be a graph, and let W be a wall in G with outer cycle D . Let us assume that there exists a separation (A, B) such that $A \cap B \subseteq V(D)$, $V(W) \subseteq B$, and every segment of W that is a subgraph of D includes a vertex of A . If some $A \cap B$ -reduction of $G[B]$ can be drawn in a disk with the vertices of $A \cap B$ drawn on the boundary of the disk in the order determined by D , then we say that the wall W is *flat in G* .

flat in G

choose the corners of W is such a way that every corner belongs to A .

We need one more definition. Given a wall W in a graph G we will (sometimes) produce a K_t minor in G . However, this K_t will not be arbitrary; it will be very closely related to the wall W . To make this notion precise we first notice that a K_t minor in G is determined by t pairwise disjoint sets X_1, X_2, \dots, X_t such that each induces a connected subgraph and every two of the sets are connected by an edge of G . We say that X_1, X_2, \dots, X_t form a *model* of a K_t minor and we will refer to the sets X_i as the *branch-sets* of the model. Often we will shorten this to a model of K_t . We say that a model of a K_t minor in G is *grasped* by a wall W if every branch-set of the model intersects at least t horizontal or at least t vertical paths of the wall. Let us remark, for those familiar with the literature, that if a wall grasps a model of K_t , then it controls it in the sense of [13]. The notion of control is important in applications, but since we were able to obtain the stronger property, we might as well state the theorem that way.

We can now formulate the Weak Structure Theorem. It first appeared in a slightly weaker form in [11, Theorem 9.8] with an unspecified bound on R in terms of t and r .

Theorem 1.5 *Let $r, t \geq 1$ be integers, let r be even, let $R = 49152t^{24}(12t^2 + r)$, let G be a graph, and let W be an R -wall in G . Then either G has a model of a K_t minor grasped by W , or there exist a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r -subwall W' of W such that $V(W') \cap A = \emptyset$ and W' is a flat wall in $G - A$.*

It is not hard to see that for $t \geq 5$ the bound on the size of A can be improved to the best possible bound of $|A| \leq t - 5$ at the expense of worsening the bound on R . We omit further details.

In Section 6 we convert the proof of Theorem 1.5 into a polynomial-time algorithm, as follows.

Theorem 1.6 *There is an algorithm with the following specifications.*

Input: *A graph G on n vertices and m edges, integers $r, t \geq 1$, and an R -wall W in G , where $R = 49152t^{24}(24t^2 + r)$.*

Output: *Either a model of K_t minor in G grasped by W , or a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r -subwall W' of W such that $V(W') \cap A = \emptyset$ and W' is a flat wall in $G - A$.*

Running time: $O(t^{24}m + t^2mn)$.

In the second alternative the algorithm also returns a separation (A, B) as in the definition of flat wall, and a certificate that the separation is as desired. The details are in the full version stated as Theorem 6.7.

1.3 The Excluded Clique Minor Theorem

As the name suggests, Theorem 1.5 is a step toward a more comprehensive excluded clique theorem of Robertson and Seymour [13].

Theorem 1.7 *For every finite graph H there exists an integer k such that every graph with no H minor can be obtained by repeated clique-sums, starting from graphs that k -near embed in a surface in which H cannot be embedded.*

Since we do not need Theorem 1.7, let us omit the precise definition of k -near embedding. Instead, let us describe it informally. A graph G can be k -near embedded in a surface Σ if there exists a set $A \subseteq V(G)$ of size at most k such that $G - A$ can be almost drawn in Σ , except for at most k areas of non-planarity, where crossings are permitted, but the graph is restricted in a different way. Here almost (similarly as in the abstract) means that we are not drawing the graph G itself, but some C -reduction instead, where now C is a large wall in G . We refer to [13] for a precise statement.

We believe that we have found a much simpler proof of Theorem 1.7 with a significantly improved bound on k . We hope to be able to report on it soon.

The paper is organized as follows. In the next three sections we prove auxiliary lemmas, and in Section 5 we prove Theorem 1.5. In Section 6 we convert the proof to a polynomial-time algorithm to construct either a K_t minor or a flat wall.

2 Disjoint M -paths with distance constraints

Let G be a graph, and let M be a subgraph of G . By an M -path we mean a path in G with at least one edge, both ends in $V(M)$ and otherwise disjoint from M . The objective of this section is to study M -paths that are “long” in the sense that their ends are at least some specified distance apart according to a metric on $V(M)$. We prove an Erdős-Pósa-type result that says that either there are many long M -paths, or all long M -paths can be destroyed by deleting a restricted set of vertices. In fact, we prove two closely related results along the same lines. It turns out that for these lemmas the distance need not be given by a metric—all that is needed is the knowledge of which pairs of vertices are far apart. We capture that using the relation R below.

Definition Let G be a graph, let M be a subgraph of G , and let R be a reflexive and symmetric relation on $V(M)$. We say that pairwise disjoint M -paths P_1, \dots, P_k are *R -semi-dispersed* if it is possible to label the ends of P_i as x_i and y_i such that $(x_i, y_i) \notin R$ and $(x_i, x_j) \notin R$ for all distinct indices $i, j \in \{1, 2, \dots, k\}$. Thus no restriction is placed on the relative position of the vertices y_1, y_2, \dots, y_k . For $x \in V(M)$ we define $R(x)$, the *ball around x* , as the set of all $y \in V(M)$ such that $(x, y) \in R$.

R -semi-dispersed

Lemma 2.1 *Let G be a graph, let M be a subgraph of G , let R be a reflexive and symmetric relation on $V(M)$, and let $k \geq 0$ be an integer. Then either there exist pairwise disjoint M -paths P_1, \dots, P_k which are R -semi-dispersed, or, alternatively, the following holds. There exist sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ with $|A| \leq k-1$ and $|Z| \leq 3k-3$ such that every M -path P in $G - A$ with ends x and y either satisfies $(x, y) \in R$ or both $x, y \in \bigcup_{z \in Z} R(z)$.*

Proof. For the duration of the proof, we will say that an M -path P is *long* if the ends x and y are long

y of P satisfy $(x, y) \notin R$. Let P_1, \dots, P_s be disjoint M -paths with the ends of P_i labeled x_i and y_i satisfying the requirements in the definition of R -semi-dispersed. Let $0 \leq p \leq s$ be an integer, and let Q_1, \dots, Q_p be disjoint paths with the ends of Q_i equal to a_i and w_i satisfying the following for all distinct integers $i, j \in \{1, 2, \dots, p\}$:

- (a) $w_i \in V(M) \setminus \bigcup_{j=1}^s (R(x_j) \cup R(y_j))$,
- (b) $a_i \in V(P_i)$,
- (c) Q_i is internally disjoint from $V(M) \cup \bigcup_{j=1}^s V(P_j)$, and
- (d) $(w_i, w_j) \notin R$.

We may assume that these paths are chosen so that s is maximum, and, subject to that, p is maximum. We may assume that $s < k$, for otherwise the first outcome of the lemma holds. We will show that the sets $A := \{a_1, a_2, \dots, a_p\}$ and $Z := \{w_1, w_2, \dots, w_p, x_1, y_1, \dots, x_s, y_s\}$ satisfy the second outcome of the lemma.

To that end let $W := \bigcup_{i=1}^p R(w_i) \cup \bigcup_{i=1}^s (R(x_i) \cup R(y_i))$. We may assume for a contradiction that there exists a long M -path S in $G - A$ which has an end in $V(M) \setminus W$. If S is disjoint from P_1, \dots, P_s , we see that S, P_1, \dots, P_s satisfy the definition of R -semi-dispersed, contrary to the maximality of s . Thus S intersects one of the paths P_i , and hence we may let y be the first vertex of $\bigcup_{i=1}^p V(Q_i) \cup \bigcup_{i=1}^s V(P_i)$ which we encounter when traversing the path S beginning at an end in $x \in V(M) \setminus W$.

There are now several different cases, depending on where the vertex y lies. As the first case, assume $y \in V(Q_i)$ for some $1 \leq i \leq p$. It follows that $S \cup Q_i$ contains a long M -path, call it P' , which has x as an end and is disjoint from P_1, \dots, P_s . Then the paths P', P_1, \dots, P_s are R -semi-dispersed, contrary to the maximality of s . As the next case, assume $y \in V(P_i)$ for some $1 \leq i \leq p$. Then $S \cup Q_i \cup P_i$ contains two disjoint long M -paths, call them P' and P'' , such that P' has x as an end and P'' has w_i as an end. Note that here we are using the property that $y \neq a_i$ to ensure that P' and P'' can be chosen disjoint. Then the paths $P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_s, P', P''$ are R -semi-dispersed, again contrary to the maximality of s . As the final case, consider when $y \in V(P_i)$ for some index i with $p < i \leq s$. We may assume, by swapping the paths P_{p+1} and P_i , that $i = p + 1$. Then the paths Q_1, \dots, Q_p, S contradict the maximality of p .

This completes the analysis of the possible cases, proving the lemma. \square

We also need the following closely related lemma. Let G be a graph, let M be a subgraph of G , and let R be a reflexive and symmetric relation on $V(M)$. We say that pairwise disjoint M -paths P_1, P_2, \dots, P_k are *R -dispersed* if $(x, y) \notin R$ for every two distinct vertices x, y such that each is an end of one of the paths P_i . R -dispersed

Lemma 2.2 *Let G be a graph, let M be a subgraph of G , let R be a reflexive and symmetric relation on M , and let $k \geq 0$ be an integer. Then either there exist pairwise disjoint R -dispersed M -paths P_1, \dots, P_k , or, alternatively, the following holds. There exist sets $A \subseteq V(G)$*

and $Z \subseteq V(M)$ with $|A| \leq k - 1$ and $|Z| \leq 3k - 3$ such that for every M -path P in $G - A$ its ends can be denoted by x and y such that either $(x, y) \in R$ or $x \in \bigcup_{z \in Z} R(z)$.

Proof. This follows by the same argument as Lemma 2.1, with the following differences. Instead of choosing the paths P_i to be R -semi-dispersed we choose them to be R -dispersed. We choose the path S to be an $(M - A - W)$ -path in $G - A$; if such a choice is not possible, then the lemma holds. We then derive a contradiction as in the proof of Lemma 2.1. \square

3 Meshes and clique minors

In this section we introduce the notion of a mesh—a common generalization of walls and grids. It will allow us to reduce problems about walls to problems about grids, which is useful, because grids are easier to work with. We also introduce a distance function on a mesh.

Definition Let $r, s \geq 2$ be positive integers, let M be a graph, and let $P_1, P_2, \dots, P_r, Q_1, Q_2, \dots, Q_s$ be paths in M such that the following conditions hold for all $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, s$:

- (1) P_1, P_2, \dots, P_r are pairwise vertex disjoint, Q_1, Q_2, \dots, Q_s are pairwise vertex disjoint, and $M = P_1 \cup P_2 \cup \dots \cup P_r \cup Q_1 \cup Q_2 \cup \dots \cup Q_s$,
- (2) $P_i \cap Q_j$ is a path, and if $i \in \{1, s\}$ or $j \in \{1, r\}$ or both, then $P_i \cap Q_j$ has exactly one vertex,
- (3) P_i has one end in Q_1 and the other end in Q_s , and when traversing P_i the paths Q_1, Q_2, \dots, Q_s are encountered in the order listed,
- (4) Q_j has one end in P_1 and the other end in P_r , and when traversing Q_j the paths P_1, P_2, \dots, P_r are encountered in the order listed.

In those circumstances we say that M is an $r \times s$ mesh. We will refer to P_1, P_2, \dots, P_r as $r \times s$ mesh *horizontal paths* and to Q_1, Q_2, \dots, Q_s as *vertical paths*. Thus every $r \times s$ grid is an $r \times s$ mesh, and, conversely, every planar graph obtained from an $r \times s$ grid by subdividing edges and splitting vertices is an $r \times s$ mesh. In particular, every r -wall is an $r \times r$ -mesh.

We wish to define a distance function on a mesh, but we first do it for a grid. Let H be the $r \times s$ grid, so that $V(H) = [r] \times [s]$. We regard H as a plane graph, using the obvious straight-line drawing. For $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ we define $d(v_1, v_2) := k - 1$, where k is the least integer such that every curve in the plane joining v_1 and v_2 intersects H at least k times. (We may clearly restrict ourselves to curves intersecting H only in vertices.) This distance can be calculated from the knowledge of the coordinates. Indeed, it is easy to check that $d(v_1, v_2)$ is equal to the minimum of $\max\{|x_1 - x_2|, |y_1 - y_2|\}$ and $\min\{x_1, y_1, r + 1 - x_1, s + 1 - y_1\} + \min\{x_2, y_2, r + 1 - x_2, s + 1 - y_2\} - 1$.

We now extend this definition to meshes as follows. Let M be a mesh with horizontal paths P_1, P_2, \dots, P_r and vertical paths Q_1, Q_2, \dots, Q_s as above. Then M has an H minor, where H is the $r \times s$ grid, as in the previous paragraph. Thus there exists a surjective mapping $f : V(M) \rightarrow V(H)$ such that $f^{-1}(u)$ is a branch-set of the H minor for every $u \in V(H)$. Furthermore, if $u = (i, j)$, then the set $f^{-1}(u)$ includes $V(P_i) \cap V(Q_j)$. If d_H denotes the distance function on H from the previous paragraph, then we define $d(u, v) := d_H(f(u), f(v))$. We say that d is a *distance function on M* . The function d is a pseudometric; that is, it is symmetric and satisfies the triangle inequality, but there may be distinct vertices u, v with $d(u, v) = 0$. The function d is not unique; it depends on the choice of the function f .

Definition The definition of grasping extends to meshes almost verbatim, as follows. We say that a model of a K_t minor in G is *grasped* by an $r \times s$ -mesh M if $t \leq \min\{r, s\}$ and every branch-set of the model intersects at least t horizontal or at least t vertical paths of the mesh. grasped

Definition Let G be a graph, let M be a mesh in G , and let \mathcal{H} and \mathcal{V} be the sets of horizontal and vertical paths of M , respectively. Let G' be a minor of G , and let \mathcal{H}' and \mathcal{V}' be the corresponding paths in G' . Let M' be a mesh in G' such that every horizontal path of M' is a subpath of a path in \mathcal{H}' and distinct horizontal paths of M' are subpaths of distinct paths in \mathcal{H}' , and every vertical path of M' is a subpath of a path in \mathcal{V}' and distinct vertical paths of M' are subpaths of distinct paths in \mathcal{V}' . In those circumstances we say that the mesh M' is *compatible with M* . compatible
with M

Lemma 3.1 *Let G be a graph, let M be a mesh in G , let G' be a minor of G , and let M' be a mesh in G' compatible with M . If for some integer $t \geq 0$ the graph M' grasps a K_t minor of G' , then M grasps a K_t minor of G .*

The proof is clear and we omit it.

Let $r \geq 1$ be an integer, and let H_{2r} be the $2r \times 2r$ -grid with vertex-set $[2r] \times [2r]$, as usual. The graph H_{2r}^1 is defined as the graph obtained from H_{2r} by adding all edges with ends (i, r) and $(i + 1, r + 1)$, and all edges with ends $(i, r + 1)$ and $(i + 1, r)$ for all $i = 1, 2, \dots, 2r - 1$. In other words, H_{2r}^1 is constructed from the $2r \times 2r$ -grid by adding a pair of crossing edges in each face of the middle row of faces. We will refer to the grid H_{2r} as the *underlying grid* of H_{2r}^1 . H_{2r}^1

Lemma 3.2 *Let $t \geq 2$ be an integer. The graph $H_{t(t-1)}^1$ has a K_t minor grasped by the underlying grid.*

Proof. The proof is by induction on t . Let the vertices of $H_{t(t-1)}^1$ be labeled as in the definition, and let L be the set of vertices of $H_{t(t-1)}^1$ with the second coordinate one. We actually prove a slightly stronger statement, to facilitate the induction. We show that $H_{t(t-1)}^1$ has a K_t minor grasped by the underlying grid such that every branch set contains a vertex in L . The statement clearly holds for $t = 2$, and so we assume that $t > 2$ and that the statement holds for $t - 1$.

Let H' be the subgraph of $H_{t(t-1)}^1$ induced by vertices (x, y) , where $1 \leq x \leq (t - 1)(t - 2)$ and $t \leq y \leq (t - 1)^2$, and let L' be the set of vertices of H' with second coordinate $(t - 1)^2$.

Then $H_{(t-1)(t-2)}^1$ is isomorphic to H' by an isomorphism that maps the first row of $H_{(t-1)(t-2)}^1$ onto L' . By the induction hypothesis the graph H' has a K_{t-1} minor with branch sets $X'_1, X'_2, \dots, X'_{t-1}$ such that $X'_i \cap L' \neq \emptyset$ for all $i = 1, 2, \dots, t-1$. Let $i \in \{1, 2, \dots, t-1\}$. Let x_i be such that $(x_i, (t-1)^2) \in X'_i \cap L'$. We may assume that $x_1 > x_2 > \dots > x_{t-1}$. We define X_i to consist of X'_i , the vertices $(x_i, (t-1)^2 + i)$, $((t-1)(t-2) + 2i - 1, (t-1)^2 + i)$, $((t-1)(t-2) + 2i - 1, t(t-1)/2 + 1)$, $((t-1)(t-2) + 2i, t(t-1)/2)$, $((t-1)(t-2) + 2i, 1)$, and the vertices of vertical and horizontal paths of the underlying grid connecting those vertices, making each X_i induce a connected subgraph of $H_{t(t-1)}^1$. Finally we define X_t as the set containing all the vertices $((t-1)(t-2) + 2i - 1, t(t-1)/2)$ and $((t-1)(t-2) + 2i, t(t-1)/2 + 1)$ for all $i = 1, 2, \dots, t-1$, and the vertices of the vertical path connecting $((t-1)(t-2) + 1, t(t-1)/2)$ to $((t-1)(t-2) + 1, 1)$. It follows that X_1, X_2, \dots, X_t are the branch sets of a K_t minor, and each branch set intersects L . This is illustrated in Figure 2. Since every X_i intersects at

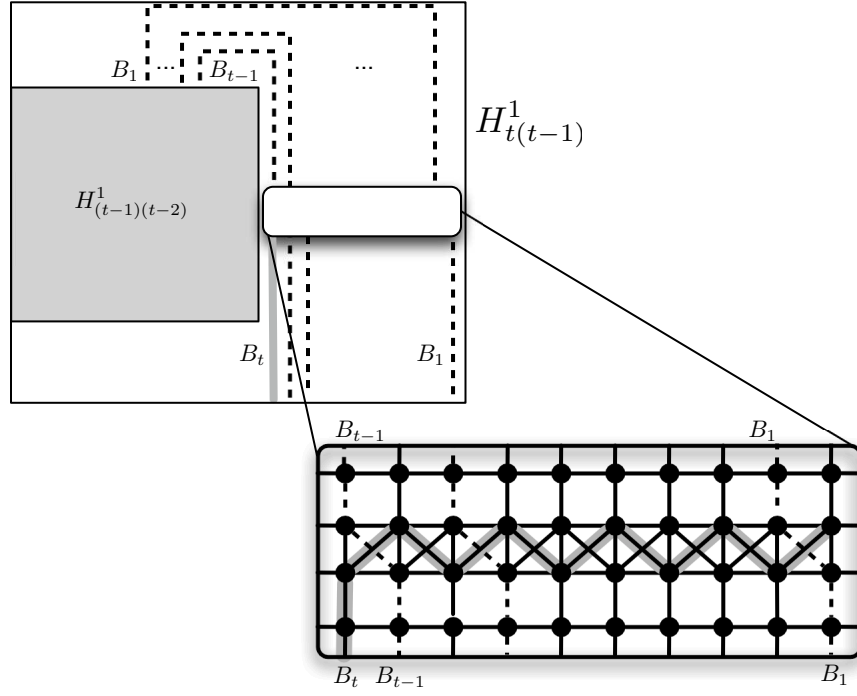


Figure 2: Finding a K_t minor in $H_{t(t-1)}^1$.

least t vertical paths of $H_{t(t-1)}^1$, we deduce that the minor is grasped by the underlying grid, as required. \square

4 Disjoint paths attaching to a mesh

The goal of this section is to show that given a mesh M in a graph G , either G has a K_t minor grasped by M , or there exist bounded number of vertices and bounded number of balls

in M of bounded radius such that after deleting those vertices and balls, every M -path has its ends close to each other.

We will need two classic lemmas going forward.

Lemma 4.1 *Let $k, r, s \geq 1$ be integers, and let \mathcal{I} be a set of k intervals on the real line. If $k \geq (r-1)(s-1) + 1$, then either \mathcal{I} has a subset of r pairwise disjoint intervals, or \mathcal{I} has a subset of s intervals that have non-empty intersection.*

Lemma 4.2 (Erdős and Szekeres) *Let $r, s \geq 1$ be integers. Every sequence of $k \geq (r-1)(s-1) + 1$ real numbers has either a non-decreasing subsequence of length r , or a non-increasing subsequence of length s .*

Lemma 4.3 *Let $t \geq 2$ be a positive integer, let $k = 32(t(t-1))^6$, let G be a graph, let M be a mesh in G with distance function d , let $X \subseteq V(M)$ with $|X| = 2k$ such that $d(x, y) \geq 2t(t-1)$ for all $x, y \in X$, and let $F \subseteq E(G) - E(M)$ be a matching of size k with vertex-set X . Then the graph G has a K_t minor grasped by M .*

Proof. The definition of distance function involves a grid minor of M . Let H be a grid minor of M that gives rise to the distance function d , and let G' be the corresponding minor of G . Then F gives rise to a matching F' in G' of size k . Given the way we defined the distance function on a mesh, the ends of the edges in F' are pairwise at distance at least $2t(t-1)$ with respect to the distance function on H . If G' has a K_t minor grasped by H , then G has a K_t minor grasped by M by Lemma 3.1. Thus it suffices to prove the lemma when M is grid.

We therefore assume for the rest of the proof that M is a grid. Let the vertices of M be labeled (x, y) for $1 \leq x \leq s$, $1 \leq y \leq r$. We number the edges in F as e_1, e_2, \dots and denote the ends of e_i by (x_i, y_i) and (u_i, v_i) . There is at most one edge of F which has an end with distance at most $t(t-1) - 1$ from a vertex of the outer cycle of M . We discard such an edge from F if it exists. The remaining edges e_i therefore satisfy

- (1) if (x, y) is an end of e_i , then $t(t-1) < x < s+1-t(t-1)$ and $t(t-1) < y < r+1-t(t-1)$.

We may temporarily assume that for every i either $x_i < u_i$, or $x_i = u_i$ and $y_i > v_i$. By reducing F to no less than half its original size we may assume that either $y_i \leq v_i$ for all i , or $y_i > v_i$ for all i . In the former case it follows that $x_i < u_i$ for all i . In the latter case we reverse the second coordinate and then swap the coordinates (formally we map each vertex (x, y) to $(r+1-y, x)$) and conclude that we may assume that for at least half the indices i

- (*) $x_i < u_i$ and $y_i \leq v_i$.

By restricting ourselves to a subset of F of size $4(t(t-1))^3$ we may assume that either $x_i \neq x_j$ for all remaining pairs of distinct edges e_i, e_j , or that $x_i = x_j$ for all such pairs. In the latter case notice that $|y_i - y_j| \geq 2t(t-1) \geq 4$, because (x_i, y_i) and (x_j, y_j) are at distance at least $2t(t-1)$. In the latter case we swap the coordinates one more time to arrive at a set $\{e_1, e_2, \dots, e_l\} \subseteq F$ such that for all distinct indices $i, j = 1, 2, \dots, l$ condition (1) holds and

- (2) either $x_i < u_i$, or $x_i = u_i$ and $|x_i - x_j| \geq 4$,
- (3) $x_i \neq x_j$, and
- (4) $l \geq 4(t(t-1))^3$.

We apply Lemma 4.1 to the set of intervals $\{[x_i, u_i] : 1 \leq i \leq l\}$. We conclude that either there exists a set $I \subseteq \{1, 2, \dots, l\}$ of size at least $t(t-1)$ such that the intervals $\{[x_i, u_i] : i \in I\}$ are pairwise disjoint, or there exist a set $J \subseteq \{1, 2, \dots, l\}$ of size at least $4(t(t-1))^2$ and an integer z such that $x_i \leq z \leq u_i$ for all $i \in J$.

Assume first that I exists. We claim that the graph obtained from M by adding the edges $\{e_i : i \in I\}$ has an $H_{t(t-1)}^1$ minor, where the underlying grid of $H_{t(t-1)}^1$ is compatible with M . To see this we use the first and last $t(t-1)$ vertical and horizontal paths of M (notice that by (1) for $i \in I$ no end of e_i belongs to any of those paths), and use the edges e_i to obtain the crossings in the middle row of faces. The i^{th} crossing will use vertices (x, y) with $t(t-1) \leq y \leq r+1-t(t-1)$ and $x_i \leq x \leq u_i$ if $x_i < u_i$ and $x_i - 1 \leq x \leq x_i + 1$ if $x_i = u_i$. Condition (2) guarantees that the crossings will be pairwise disjoint. By Lemma 3.2 the graph $H_{t(t-1)}^1$ has a K_t minor grasped by the underlying grid of $H_{t(t-1)}^1$. By Lemma 3.1 the graph G has a K_t minor grasped by M , as desired. This completes the case when I exists.

We may therefore assume that J and z exist. By renumbering the indices we may assume that $x_1 < x_2 < \dots < x_{4(t(t-1))^2} < z$ and $u_i \geq z$ for all $1 \leq i \leq 4(t(t-1))^2$. Let M_1 be the subgraph of M induced by vertices (x, y) with $1 \leq x < z$ and $1 \leq y \leq r$, and let M_2 be the subgraph of M induced by vertices (x, y) with $z \leq x \leq s$ and $1 \leq y \leq r$. We see that $(u_i, v_i) \in V(M_2)$ for all $1 \leq i \leq 4(t(t-1))^2$. Let P be a path in M_2 covering the vertices of M_2 . The edges e_i for $1 \leq i \leq 4(t(t-1))^2$ each have one end in P and one end in $V(M_1)$. By Lemma 4.2 there exists a sequence $1 \leq i_1 < i_2 < \dots < i_{2t(t-1)}$ such that the ends of $e_{i_1}, e_{i_2}, \dots, e_{i_{2t(t-1)}}$ occur on P in the order listed. For $j = 1, 2, \dots, t(t-1)$ we make use of the edges $e_{i_{2j-1}}, e_{i_{2j}}$ and the subpath of P connecting the ends of $e_{i_{2j-1}}$ and $e_{i_{2j}}$ to construct an M_1 -path with ends $x_{i_{2j-1}}$ and $x_{i_{2j}}$. The paths just constructed are pairwise vertex-disjoint, and, similarly as in the previous paragraph, can be used to deduce that G has an $H_{t(t-1)}^1$ minor, where the underlying grid is compatible with M_1 , and hence with M . By Lemma 3.2 the graph $H_{t(t-1)}^1$ has a K_t minor grasped by the underlying grid of $H_{t(t-1)}^1$. By Lemma 3.1 the graph G has a K_t minor grasped by M , as desired. \square

Lemma 4.4 *Let B be a connected graph of maximum degree at most four, and let $Y \subseteq V(B)$. Then there exist at least $(|Y| - 1)/4$ disjoint paths in B , each with at least one edge and with both ends in Y .*

Proof. We may assume for a contradiction that the conclusion does not hold and, subject to that, $|E(B)|$ is minimum. By a *leaf* of B we mean a vertex of degree one. Then B is a tree, every leaf belongs to Y , $|Y| \geq 6$ and (by contracting an incident edge we see that) every vertex of degree two belongs to Y . Let L be the set of leaves of B . Since $|Y| \geq 3$ the graph $B - L$ is a tree, and therefore we may select a leaf t of $B - L$. Since t does not have degree two in B , it is incident with least two leaves of B , say t_1 and t_2 . On the other hand, since

t has degree at most four and $|Y| \geq 6$, the vertex t is adjacent to at most three leaves of B . Let B' be the graph obtained from B by deleting t and all leaves adjacent to it, and let $Y' := Y \cap V(B')$. By the minimality of B there exist at least $(|Y'| - 1)/4 \geq (|Y| - 1)/4 - 1$ disjoint paths in B' , each with at least one edge and both ends in Y' . By adding the path with vertex-set $\{t_1, t, t_2\}$ we obtain a collection as required in the lemma, a contradiction. \square

Before the next lemma, let us remark that $3 \times 2^{12} = 12288$.

Lemma 4.5 *Let $t \geq 1$ be an integer, let $k_0 := 12288(t(t-1))^{12}$, let G be a graph, and let M be a mesh in G with distance function d . Then either G has a K_t minor grasped by M , or there exist sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ such that $|A| \leq k_0 - 1$, $|Z| \leq 3k_0 - 3$, and if x, y are the ends of an M -path in $G - A$, then either $d(x, y) < 6t(t-1)$, or each of x, y lies at distance at most $6t(t-1) - 1$ from some vertex of Z .*

Proof. Let us define a relation R on $V(M)$ by saying that $(x, y) \in R$ if $d(x, y) < 2t(t-1)$. By Lemma 2.2 applied to the relation R , graph M and integer $k = 32(t(t-1))^6$ we deduce that one of the two outcomes holds. If the first outcome holds, then G has K_t minor grasped by M by Lemma 4.3, and hence our lemma holds. Thus we may assume that the second outcome of Lemma 2.2 holds, and hence there exist sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ with $|A| \leq k - 1$ and $|Z| \leq 3k - 3$ such that

- (1) *for every M -path P in $G - A$ its ends can be denoted by x and y such that either $d(x, y) < 2t(t-1)$ or $d(x, z) < 2t(t-1)$ for some $z \in Z$.*

We now change the definition of R to mean that $(x, y) \in R$ if $d(x, y) < 6t(t-1)$ and apply Lemma 2.1 to the relation R , graph M and integer k_0 . If the second outcome holds, then the second outcome of the current lemma holds, and so we may assume that the first outcome of Lemma 2.1 holds. Thus there exists a set \mathcal{P}_2 of k_0 pairwise disjoint M -paths that are R -semi-dispersed. Thus the ends of every path $P \in \mathcal{P}_2$ can be denoted by $x(P)$ and $y(P)$ such that $x(P)$ and $y(P)$ are at distance at least $6t(t-1)$ for every P , and $x(P)$ and $x(P')$ are at distance at least $6t(t-1)$ for every two distinct paths $P, P' \in \mathcal{P}_2$. The set \mathcal{P}_2 has a subset of size at least $k_0 - k$ such that each member is disjoint from A . By (1) there exists $z \in Z$ and a subset of the latter set of paths of size at least $(k_0 - k)/(3k - 3)$ such that every member P of the latest set has the property that one of $x(P), y(P)$ is at distance at most $2t(t-1)$ from z . Let B denote the subgraph of M induced by vertices of M at distance at most $2t(t-1)$ from z . Since the vertices $x(P)$ are pairwise at distance at least $6t(t-1)$, we deduce that $x(P) \in V(B)$ for at most one of those paths P . By omitting that path we obtain a set $\mathcal{P}_3 \subseteq \mathcal{P}_2$ of disjoint M -paths in $G - A$ with $y(P) \in V(B)$ for every $P \in \mathcal{P}_3$ and such that \mathcal{P}_3 has cardinality at least $(k_0 - k)/(3k - 3) - 1 \geq 128(t(t-1))^6$.

Let \mathcal{Q} be the set of vertical and horizontal paths of M that are disjoint from B . We wish to define a mesh M' consisting of subpaths of members of \mathcal{Q} . The definition is as follows. Let P_1, P_2, \dots, P_r be the vertical paths of M , let Q_1, Q_2, \dots, Q_s be the horizontal paths of M , and let I, J be such that \mathcal{Q} consists of P_i and Q_j for all $i \in I$ and all $j \in J$. Let $i_0 := \min I$, $i_1 := \max I$, $j_0 := \min J$, and $j_1 := \max J$. For $i \in I$ let P'_i be the shortest subpath of P_i

from Q_{j_0} to Q_{j_1} , and for $j \in J$ let Q'_j be the shortest subpath of Q_j from P'_{i_0} to P'_{i_1} . Let M' be the union of P'_i and Q'_j for all $i \in I$ and $j \in J$. It is not hard to see that M' is a mesh. We now select a distance function on M' as follows. Let H be a grid minor of M that gave rise to the distance function d on M , as in the definition of distance function. Starting with M' we first contract all edges that were contracted during the production of H from M , and then contract edges arbitrarily until we arrive at a grid H' . We use H' in order to define a distance function d' on M' . It follows that

$$(2) \quad d'(x, y) \geq d(x, y) - 4t(t-1) \text{ for all } x, y \in V(M').$$

Let $P \in \mathcal{P}_3$, and let $x = x(P)$. We wish to define a path $\phi(P)$ with one end x . If $x \in V(M')$, then $\phi(P)$ is defined to be the path with vertex-set $\{x\}$; otherwise we proceed as follows. By symmetry between the paths P_i and Q_j we may assume that $x \in V(P_i)$. We claim that $P_i \notin \mathcal{Q}$. To prove this claim suppose to the contrary that $P_i \in \mathcal{Q}$. Since $x \notin V(M')$ it follows that when traversing P_i starting from Q_0 we either encounter x strictly before Q_{i_0} , or we encounter x strictly after Q_{j_0} . In either case it follows that $x \in V(B)$, a contradiction. This proves our claim that $P_i \notin \mathcal{Q}$. Let j be such that either $x \in V(Q_j)$, or when traversing P_i as above we encounter Q_j , then x , and then Q_{j+1} . Then at least one of Q_j, Q_{j+1} belongs to \mathcal{Q} , for otherwise $x \in V(B)$, a contradiction (if $x \in V(Q_j)$, then $Q_j \in \mathcal{Q}$). If $Q_j \in \mathcal{Q}$, then let $\phi(P)$ be the shortest subpath of P_i from x to $x' \in V(Q_j)$; otherwise let $\phi(P)$ be the shortest subpath of P_i from x to $x' \in V(Q_{j+1})$. The argument used above to show that $P_i \notin \mathcal{Q}$ now implies that $x' \in V(M')$.

Let Y be the set of all vertices $y(P)$ over all paths $P \in \mathcal{P}_3$. Since the graph B is connected, by Lemma 4.4 there exists a set \mathcal{R} of at least $\lceil (|\mathcal{P}_3| - 1)/4 \rceil \geq 32(t(t-1))^6$ disjoint subpaths of B , each with distinct ends in Y . For each $R \in \mathcal{R}$ with ends y_1 and y_2 we define an M' -path by taking the union $R \cup P_1 \cup \phi(P_1) \cup P_2 \cup \phi(P_2)$, where $P_i \in \mathcal{P}_3$ satisfies $y(P_i) = y_i$. These paths are pairwise vertex-disjoint. Since for distinct paths $P, P' \in \mathcal{P}_3$ the vertices $x(P), x(P')$ are at distance at least $6t(t-1)$ in M , they are at distance at least $2t(t-1)$ in M' by (2). By Lemma 4.3 the graph G has a K_t minor grasped by M' , and hence it has a K_t minor grasped by M by Lemma 3.1, as desired. \square

5 Proof of the Weak Structure Theorem

We need a lemma before we can begin the proof of Theorem 1.5.

Lemma 5.1 *Let G be a graph, let W be a subgraph of G , let D be an induced cycle in W , and let C be a cycle in G such that $W - V(D)$ is connected and there exist four internally disjoint paths from $V(W) - V(D)$ to $V(C)$ such that each intersects D . If some C -reduction of G can be drawn in the plane with C bounding a face, then there exists a separation (A, B) in G such that*

$$(1) \quad A \cap B \subseteq V(D),$$

- (2) $V(W) \subseteq B$,
- (3) $V(C) \subseteq A$, and
- (4) *some $A \cap B$ -reduction of $G[B]$ can be drawn in a disk with $A \cap B$ drawn on the boundary of the disk in the order determined by D .*

Proof. We proceed by induction on the number of vertices of G . Let us first assume that G itself can be drawn in the plane with C bounding a face. Let us consider a planar drawing of G in which C bounds the outer face, let B be the set of vertices drawn in the closed disk bounded by D , and let A be the set of vertices drawn in the complement of its interior. Then (A, B) is a separation of G with $A \cap B = V(D)$ and it clearly satisfies (2) and (4). We claim that $V(W) \subseteq B$. To prove this claim we may assume for a contradiction that some vertex of W belongs to $A - B$. Since $W - V(D)$ is connected, this implies that $V(W) \subseteq A$. Let G' be obtained from $G[A]$ by adding a new vertex in the open disk bounded by D and joining it by an edge to every vertex of D . Then G' is planar, and yet by considering $V(W) - V(D)$, $V(C)$, three of the four disjoint paths between them, and the new vertex, we find that G' has a $K_{3,3}$ subdivision, a contradiction. This proves our claim that $V(W) \subseteq B$, and hence (A, B) satisfies (1)–(4). This completes the case when G itself can be drawn in the plane with C bounding a face.

We may therefore assume that G has an elementary C -reduction G' on strictly fewer vertices such that some C -reduction of G' can be drawn in the plane with C bounding the outer face. Let (X, Y) be the separation of G that determines G' ; that is, $|X \cap Y| \leq 3$, $V(C) \subseteq X$, there exist $|X \cap Y|$ disjoint paths from some vertex of Y to $V(C)$, and G' is obtained from $G[X]$ by adding all edges with both ends in $X \cap Y$. Since there are four internally disjoint paths from $V(W) - V(D)$ to $V(C)$ it follows that $V(W) - V(D) \not\subseteq Y$, and since each of them intersects $V(D)$ we conclude that $V(D) \not\subseteq Y$. Let W' be obtained from $W[X]$ by adding an edge joining every pair of vertices in $X \cap Y$, and let D' be obtained from D by replacing each subpath of D with both ends in $X \cap Y$ and all internal vertices in Y by the new edge joining its ends. Then D' is a cycle in W' and $W' - V(D')$ is non-null and connected. Since $W - V(D)$ is connected, the four internally disjoint paths P_1, P_2, P_3, P_4 from $W - V(D)$ to $V(C)$ that intersect $V(D)$ may be chosen so that P_1, P_2 originate in say $x \in V(W) - V(D)$, P_3, P_4 originate in say $y \in V(W) - V(D)$, and there exists a path P_0 in $W - V(D)$ with ends x, y , such that the paths P_0, P_1, \dots, P_4 are pairwise disjoint, except when x or y are a common end. With this observation it is now easy to construct four internally disjoint paths from $W' - V(D')$ to $V(C)$ such that each intersects $V(D')$. By the induction hypothesis there exists a separation (A', B') of G' that satisfies (1)–(4) when G, A, B, W, D are replaced by G', A', B', W', D' , respectively.

Since the vertices of $X \cap Y$ are pairwise adjacent in G' we deduce that either $X \cap Y \subseteq A'$ or $X \cap Y \subseteq B'$. In the former case we define $A := A' \cup Y$ and $B := B' \cup (V(D) \cap Y)$; in the latter case we define $A := A'$ and $B := B' \cup Y$. In either case (A, B) is a separation of G , and we claim that it satisfies (1)–(4). Conditions (1) and (3) are clear, and so it remains to prove (2) and (4).

To prove that (A, B) satisfies (2) we first notice that $V(D) \subseteq B$ by the definition of B and the fact that (A', B') satisfies (2). We will now show that $V(W) - V(D) \subseteq B$. Since

$\emptyset \neq V(W') \setminus V(D') \subseteq B'$, we deduce that $(V(W) \setminus V(D)) \cap B \neq \emptyset$. Since $W - V(D)$ is connected and disjoint from $A \cap B$, it follows that $V(W) \subseteq B$, as desired. This proves that (A, B) satisfies (2).

To prove that (A, B) satisfies (4) we know, since (A', B') satisfies (4), that some $A' \cap B'$ -reduction of $G'[B']$ can be drawn in a disk with $A' \cap B'$ drawn on the boundary of the disk in the order determined by D' . If $X \cap Y \subseteq B'$, then $A \cap B = A' \cap B'$ and $G'[B']$ is an elementary $A \cap B$ -reduction of $G[B]$, and hence (4) follows. We may therefore assume that $X \cap Y \subseteq A'$. Then $G[B]$ may be obtained from $G'[B']$ by adding zero, one, or two subpaths P of D such that P has both ends in $X \cap Y$, each internal vertex in $Y - X$, and the ends of P are consecutive in the cyclic order on $A' \cap B'$. It is now straightforward to modify the above $A' \cap B'$ -reduction of $G'[B']$ to serve as an $A \cap B$ -reduction of $G[B]$ that can be drawn in a disk with $A \cap B$ drawn on the boundary of the disk in the order determined by D . \square

Let H be a subgraph of a graph G . An H -bridge in G is a connected subgraph B of G such that $E(B) \cap E(H) = \emptyset$ and either $E(B)$ consists of a unique edge with both ends in H , or for some component C of $G \setminus V(H)$ the set $E(B)$ consists of all edges of G with at least one end in $V(C)$. The vertices in $V(B) \cap V(H)$ are called the *attachments* of B .

H -bridge

attachments

We are now ready to prove the Weak Structure Theorem, which we restate.

Theorem 5.2 *Let $r, t \geq 1$ be integers, let r be even, let $R = 49152t^{24}(12t^2 + r)$, let G be a graph, and let W be an R -wall in G . Then either G has a model of a K_t minor grasped by W , or there exist a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r -subwall W' of W such that $V(W') \cap A = \emptyset$ and W' is a flat wall in $G - A$.*

Proof. Let $t, r \geq 1$, J , and W be given, where W is an R -wall in G , and $R \geq 4 \cdot 12288t^{24}(12t(t-1) + r)$. Let d be a distance function on W . By Lemma 4.5 applied to the mesh W and distance function d we may assume that there exist sets $A \subseteq V(G)$ and $Z \subseteq V(W)$ such that

- (1) $|A| \leq 12288(t(t-1))^{12}$, $|Z| \leq 3 \cdot 12288(t(t-1))^{12}$, and if x, y are the ends of a W -path in $G - A$, then either $d(x, y) < 6t(t-1)$, or each of x, y lies at distance at most $6t(t-1) - 1$ from some vertex of Z .

Let the horizontal paths of W be P_0, \dots, P_R and the vertical paths Q_0, \dots, Q_R . A *strip* of W is a subgraph of W consisting of $12t(t-1) + r$ consecutive horizontal paths of W , say $P_{i+1}, \dots, P_{i+12t(t-1)+r}$, along with every subpath Q of a vertical path of W such that Q has both ends in $V(P_{i+1}) \cup \dots \cup V(P_{i+12t(t-1)+r})$. By our choice of R , there exists a strip S consisting of paths numbered as above such that $t(t-1)/2 < i < R - 25t(t-1)/2 - r$ and S contains no vertex of $Z \cup A$. Thus the condition on i guarantees that the strip S is surrounded by at least $t(t-1)/2$ horizontal paths on either side. We conclude that there exist subwalls $W_1, \dots, W_{t(t-1)}$ contained in S satisfying the following for all distinct integers $i, j = 1, 2, \dots, t(t-1)$:

- (2) W_i is a $12t(t-1) + r$ -wall such that the horizontal paths of W_i are subpaths of the horizontal paths of the strip S and the vertical paths of W_i are subpaths of the vertical paths of S , and
- (3) if $x \in V(W_i)$ and $y \in V(W_j)$, then $d(x, y) \geq 6t(t-1)$.

See Figure 3.

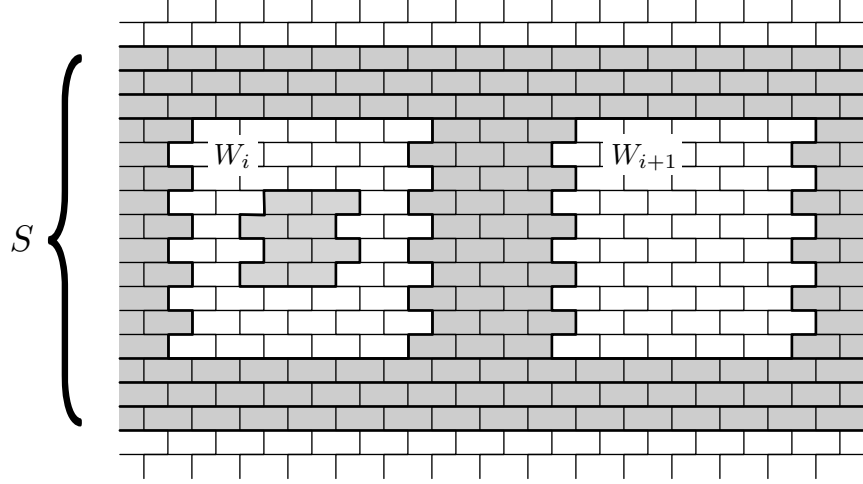


Figure 3: Subwalls of a strip.

For $i = 1, 2, \dots, t(t-1)$ we define a graph H_i . Let us recall that the corners of a wall were defined at the end of the first paragraph of Subsection 1.2. Let C_i be a cycle with vertex-set the four corners of the wall W_i in the order of their appearance on the outer cycle of W_i . In other words, the cycle C_i may be obtained from the outer cycle of W_i by suppressing all vertices, except the four corners of W_i . Let B be a $W-A$ -bridge in the graph $G-A$ with at least one attachment in $V(W_i)$, and let B' be obtained from B by deleting all its attachments that do not belong to $V(W_i)$. The graph H_i is defined as the union of the wall W_i , the cycle C_i and all graphs B' as above. We claim that the subgraphs H_i are pairwise disjoint. To see this, if there exist indices i and j with $i \neq j$ such that H_i and H_j share a vertex, then there exists a $(W-A)$ -bridge with an attachment $x \in V(W_i)$ and $y \in V(W_j)$. However then there exists a W -path with ends x and y , contrary to (1) and (3).

If for all $i = 1, 2, \dots, t(t-1)$ the graph H_i has a C_i -cross, then, by the condition imposed on the index on the first path of the strip, the graph G has a $H_{t(t-1)}^1$ minor such that the underlying grid is compatible with the original wall W . By Lemma 3.2 the graph $H_{t(t-1)}^1$ has a K_t minor grasped by the underlying grid of $H_{t(t-1)}^1$, and hence G has a K_t minor grasped by W by Lemma 3.1, as desired.

We conclude that we may assume that there exists an index i such that the graph H_i does not have a C_i -cross. By Theorem 1.3 some C_i -reduction of H_i can be drawn in the plane with

C_i bounding a face. Let W' be the r -wall obtained from W_i by deleting the first and final $6t(t-1)$ of both the horizontal and vertical paths of W_i , and let D be the outer cycle of W' . By Lemma 5.1 applied to the graph H_i , wall W' , cycle D in W' and the cycle C_i there exists a separation (X', Y) of H_i satisfying (1)–(4) of Lemma 5.1. Let $X := X' \cup (V(G) \setminus A \setminus V(H_i))$. Then $X \cap Y = X' \cap Y \subseteq V(D)$, $V(W') \subseteq V(Y)$, $V(C_i) \subseteq X$, and some $X \cap Y$ -reduction of $G[Y]$ can be drawn in a disk with $X \cap Y$ drawn on the boundary of the disk.

We claim that (X, Y) is a separation of $G - A$. To prove this claim suppose for a contradiction that $x \in X \setminus Y$ is adjacent to $y \in Y \setminus X$. Then $y \in V(H_i)$ and $x \notin V(H_i)$, because (X', Y) is a separation of H_i . We have $y \notin V(W_i) \setminus V(W')$, for otherwise $W_i - V(W')$ includes a path from y to $V(C_i)$ disjoint from $V(D)$, contrary to the facts that $V(C_i) \subseteq X'$, $y \in Y$ and $X' \cap Y \subseteq V(D)$. It follows that the edge joining x and y belongs to a $W - A$ -bridge of $G - A$, and hence x is an attachment of that $W - A$ -bridge outside W_i . It follows that this $W - A$ -bridge includes a W -path with one end x and the other end say $x' \in V(W')$. It follows that x' is at distance at least $6t(t-1)$ from x and every vertex in Z , contrary to (1). This proves that (X, Y) is a separation of G .

Let L be a segment of W that is a subgraph of D . We must show that L includes a vertex of X . To that end we may select a path P in W with one end in C_i , the other end say $l \in V(L)$, and otherwise disjoint from D . It follows that $V(P) \setminus \{l\} \subseteq X - Y$, and hence $l \in X$, as desired.

Thus the separation (X, Y) is a witness that W' is a flat wall in $G - A$. We have $V(W') \cap A = \emptyset$, because W' is a subgraph of the strip S , and S was chosen disjoint from A . \square

6 An Algorithm

We need algorithmic versions of Lemmas 2.1 and 2.2. In order for those algorithms to run efficiently we need to make some assumptions about the computability of the relation R . It seems best to do so in the context of our application, namely when M is a mesh in the graph G and $(x, y) \in R$ if and only if $d(x, y) < l$ for some integer l , where d is a distance function on M . Let us recall that the notion of a distance function was defined at the beginning of Section 3 by saying that $d(x, y)$ is the distance of $f(x)$ and $f(y)$ in H , where H is a grid minor of M and $f : V(M) \rightarrow V(H)$ describes the contraction. We will refer to $f : V(M) \rightarrow V(H)$ as a *grid contraction function*. It is clear that given a grid contraction function f , the value $d(x, y)$ can be computed in constant time for any $x, y \in V(M)$. Thus we will use a grid contraction function to represent the distance function on M . We assume that for each $x \in V(M)$ we store the value $f(x)$, and that for each $u \in V(H)$ we store $f^{-1}(u)$ as a list.

Let an integer $l \geq 0$ be fixed, and let $(x, y) \in R$ if and only if $d(x, y) < l$. We need to clarify one issue about the sets $R(x)$. Let us recall that $R(x)$ denotes the set of all $y \in X$ such that $(x, y) \in R$. If $x \in V(M)$, then $R(x)$ can be written as $\bigcup_{v \in V_1 \cup V_2} f^{-1}(v)$ for some sets $V_1, V_2 \subseteq V(H)$, where $|V_1| \leq (2l-1)^2$ and V_2 is the union of the vertex-sets of at most $2l-1$ vertical and at most $2l-1$ horizontal paths of H . To see this let V_1 be the set of all vertices $v \in V(H)$ such that there is a curve in the plane connecting v and $f(x)$ that intersects H at

most l times and does not use the outer face of H , and V_2 is defined analogously using curves that use the outer face of H .

The following is an algorithmic version of Lemma 2.1. The conclusion is slightly weaker in order to save on running time.

Lemma 6.1 *There exists an algorithm with the following specifications.*

Input: *A graph G on n vertices and m edges, integers $k, l \geq 1$, and a mesh M in G with grid contraction function $f : V(M) \rightarrow V(H)$ giving rise to a distance function d on M . For $x, y \in V(M)$ let $(x, y) \in R$ if and only if $d(x, y) < l$.*

Output: *Either k disjoint R -semi-dispersed M -paths, or sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ with $|A| \leq k - 1$ and $|Z| \leq 3k - 3$ such that every M -path P in $G - A$ with ends x and y either satisfies $d(x, y) \leq 2l - 2$ or both $x, y \in \bigcup_{z \in Z} R(z)$.*

Running time: $O(km + n)$.

Proof. We may assume that G has no isolated vertices (by deleting them). If l is at least the number of vertical or horizontal paths in M , then $A := \emptyset$ and any one-element set $Z \subseteq V(M)$ (or $Z = \emptyset$ if $k = 1$ and no M -path with ends far apart exists) satisfy the second condition of the output requirement. Thus we may assume that $l^2 = O(m)$.

The algorithm will proceed in at most $3k$ iterations. At the beginning of each iteration there will be M -paths P_1, P_2, \dots, P_s and Q_1, Q_2, \dots, Q_p as in the proof of Lemma 2.1 with ends denoted in the same way. Let A, Z, W be defined as in the proof of Lemma 2.1. At the start of the first iteration we have $s = p = 0$; thus $A = Z = W = \emptyset$. Throughout the algorithm the set W will be of the form $\bigcup_{v \in V} f^{-1}(v)$ for some $V \subseteq V(H)$, and will be presented by marking the elements of V .

For the purpose of this paragraph and the next let us say that a *good path* is an M -path S in $G - A$ with ends x, y , where $x \in V(M) - W$ and $(x, y) \notin R$. We say that S is *very good* if it is good and $d(x, y) \geq 2l - 1$. At the beginning of each iteration we either find a good path, or establish that no very good path exists. We do so by running the following subroutine for every M -bridge B of the graph $G - A$. In the subroutine we first test whether B has an attachment $x \in V(M) - W$. If not, then B does not include a good path and we return that information. Otherwise we test whether B has an attachment y at distance at least l from x ; if we find one, then a path in B from x to y is a good path, and we return it. On the other hand, if all attachments of B belong to $R(x)$, then B includes no very good path, and we return that information. This completes the description of the subroutine. It is clear that each call takes time $O(|E(B)|)$, and that if no call to the subroutine returns a good path, then no very good path exists. Thus we either find a good path, or establish that no very good path exists in time $O(m)$.

If no very good path exists, then the sets A and Z satisfy the specifications of the algorithm. We output those sets and terminate the algorithm. If we find a good path S , then we modify the paths P_i and Q_i as in the proof of Lemma 2.1 by either adding a new path P_{s+1} and keeping the old paths Q_i , or by adding two new paths P_{s+1}, P_{s+2} and discarding one old path Q_i , or by adding a new path Q_{p+1} . In each case the quantity $2s + p$ increases by one. We update the sets A, Z and W . The set W will be updated by marking $f(v)$ for every vertex

v that is being added to W . For every vertex that is being added to Z this involves marking at most $(2l-1)^2$ vertices of H and the vertex-sets of at most $2(l-1)$ vertical and at most $2(l-1)$ horizontal paths of H . The marking of vertical and horizontal paths will be done implicitly, so that the total time spend on marking during each iteration will be $O(l^2)$. If $s \geq k$ we output the paths P_1, P_2, \dots, P_k and terminate the algorithm; otherwise we go to the next iteration. The second step of the iteration described in this paragraph takes time $O(l^2 + n) = O(m)$.

Since the quantity $2s+p$ increases during each iteration and $p \leq s$, the algorithm will terminate after at most $3k$ iterations. Thus the running time is as claimed. \square

Likewise there is a version of Lemma 2.2 with a similar proof, which we omit.

Lemma 6.2 *There exists an algorithm with the following specifications.*

Input: A graph G on n vertices and m edges, integers $k, l \geq 0$, and a mesh M in G with grid contraction function $f : V(M) \rightarrow V(H)$ giving rise to a distance function d on M . For $x, y \in V(M)$ let $(x, y) \in R$ if and only if $d(x, y) < l$.

Output: Either k disjoint R -dispersed M -paths, or sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ with $|A| \leq k-1$ and $|Z| \leq 3k-3$ such that for every M -path P in $G-A$ its ends can be denoted by x and y such that either $d(x, y) \leq 2l-2$ or $x \in \bigcup_{z \in Z} R(z)$.

Running time: $O(km + n)$.

Lemma 6.3 *There is an algorithm with the following specifications.*

Input: A graph G on n vertices and m edges, an integer $t \geq 2$, a mesh in G with grid contraction function $f : V(M) \rightarrow V(H)$ giving rise to a distance function d on M , a set $X \subseteq V(M)$ with $|X| = 64(t(t-1))^6$ such that $d(x, y) \geq 2t(t-1)$ for all $x, y \in X$, and a matching $F \subseteq E(G) - E(M)$ in G of size $32(t(t-1))^6$ with vertex-set X .

Output: A model of K_t grasped by M .

Running time: $O(m + n)$.

Proof. This follows from the proof of Lemma 4.3, because it is easy to convert the standard proofs of Lemmas 4.1 and 4.2 into algorithms with running times $O(k^2) = O(m)$. \square

Lemma 6.4 *There exists an algorithm with the following specifications.*

Input: A graph G on n vertices and m edges, an integer $t \geq 2$, and a mesh in G with grid contraction function $f : V(M) \rightarrow V(H)$ giving rise to a distance function d on M .

Output: For $k_0 := 12288(t(t-1))^{12}$ either a model of K_t in G grasped by M , or sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ such that $|A| \leq k_0-1$, $|Z| \leq 3k_0-2$, and if x, y are the ends of a M -path in $G-A$, then either $d(x, y) < 12t(t-1)$, or each of x, y lies at distance at most $6t(t-1)-1$ from some vertex of Z .

Running time: $O(t^{24}m + n)$

Proof. The algorithm follows the proof of Lemma 4.5. We first apply the algorithm of Lemma 6.2 to the graph G , mesh M and integers $l = 2t(t-1)$ and $k = 32(t(t-1))^6$. If the algorithm returns k disjoint dispersed M -paths, then we use the algorithm of Lemma 6.3 to

output a model of K_t grasped by M and stop. We may therefore assume that the algorithm of Lemma 6.2 returns sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ such that $|A| \leq k - 1$, $|Z| \leq 3k - 3$, and for every M -path P in $G - A$ its ends may be denoted by x and y such that either $d(x, y) \leq 4t(t - 1) - 2$ or $d(x, z) \leq 2t(t - 1) - 1$ for some $z \in Z$. Next we apply the algorithm of Lemma 6.1 to the graph G , mesh M and integers $l = 6t(t - 1)$ and k_0 . If the algorithm returns sets A and Z , then we return those sets and stop. We may therefore assume that the algorithm of Lemma 6.1 returns a set of k_0 pairwise disjoint semi-dispersed M -paths. We use the argument of the proof of Lemma 4.5 to use the paths to construct a matching to which we can apply the algorithm of Lemma 6.3 to output a model of K_t grasped by M . \square

To state an algorithmic version of Theorem 1.3 we need a definition. Let G be a graph, let C be a cycle in G , and let J be a C -reduction of G obtained by successively performing elementary C -reductions determined by separations $(A_1, B_1), (A_2, B_2), \dots, (A_k, B_k)$. We say that (B_1, B_2, \dots, B_k) is a C -reduction sequence for G leading to J . The following is an algorithm of Shiloach [16] stated using our terminology. C-reduction sequence

Theorem 6.5 *There is a polynomial-time algorithm with the following specifications.*

Input: A graph G with n vertices and m edges and a cycle C in G .

Output: Either a C -cross in G , or a C -reduction sequence leading to a C -reduction J , and a drawing of J in the plane with C bounding the outer face.

Running time: $O(nm)$.

Our last lemma is an algorithmic version of Lemma 5.1.

Lemma 6.6 *There exists an algorithm with the following specifications.*

Input: A graph G on n vertices and m edges, a subgraph W of G , an induced cycle D in W , a cycle C in G such that $W - V(D)$ is connected, four internally disjoint paths from $V(W) - V(D)$ to $V(C)$ such that each intersects D , a C -reduction sequence Y_1, Y_2, \dots, Y_k in G leading to a C -reduction J , and drawing of J in the plane with C bounding a face.

Output: A separation (A, B) in G satisfying (1)–(4) of Lemma 5.1.

Running time: $O(n + m)$.

Proof. For $i = 1, 2, \dots, k$ let $X_i \subseteq V(G)$ be such that $(X_1, Y_1), (X_2, Y_2), \dots, (X_k, Y_k)$ are the separations that determine the C -reduction J . We will not compute the sets X_i , but we will compute $Z_i := X_i \cap Y_i$. Let P_1, P_2, P_3, P_4 be four internally disjoint paths from $V(W) - V(D)$ to $V(C)$ such that each intersects D . Furthermore, we may select them in such a way that P_1, P_2 originate in $x \in V(W) - V(D)$, the paths P_3, P_4 originate in $y \in V(W) - V(D)$, and there exists a path P_0 in $W - V(D)$ with ends x and y such that the paths P_0, P_1, \dots, P_4 are pairwise disjoint, except when x or y are a common end. For $i = 1, 2, \dots, k$ we delete $Y_i \setminus Z_i$, add all edges with both ends in Z_i , and modify $M, W, D, P_0, P_1, \dots, P_4$ as in the proof of Lemma 5.1. Thus we arrive at a graph G isomorphic to J , at which point we construct a separation (A, B) of G using the cycle D as in the proof of Lemma 5.1. Then for $i = k, k - 1, \dots, 1$ we adjust the separation (A, B) as in the proof of Lemma 5.1 so that in the end the resulting separation will satisfy the required properties in the original graph G . Finding the original separation takes time $O(m)$, and for $i = 1, 2, \dots, k$ the adjustments take time $O(|E(G[Y_i])|)$. Thus the total running time is $O(m)$. \square

We are finally ready to describe our main algorithm.

Theorem 6.7 *There is an algorithm with the following specifications.*

Input: *A graph G on n vertices and m edges, integers $r, t \geq 1$, and an R -wall W in G , where $R = 49152t^{24}(24t^2 + r)$.*

Output: *Either a model of a K_t minor in G grasped by W , or a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r -subwall W' of W such that $V(W') \cap A = \emptyset$ and W' is a flat wall in $G - A$. In the second alternative the algorithm also returns a separation (A, B) as in the definition of flat wall, an $A \cap B$ -reduction sequence for $G[B]$ leading to a $A \cap B$ -reduction J of $G[B]$, and a drawing of J in the unit disk with the vertices $A \cap B$ drawn on the boundary of the disk in the order determined by the outer cycle of W' .*

Running time: $O(t^{24}m + t^2mn)$.

Proof. We compute a grid contraction function $f : V(W) \rightarrow V(H)$ and apply the algorithm of Lemma 6.4 to the graph G , mesh W , function f , and integer t . If the algorithm returns a model of K_t grasped by W , then we return that model and stop. We may therefore assume that the algorithm of Lemma 6.4 returned sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ such that $|A| \leq 12288(t(t-1))^{12}$, $|Z| \leq 3 \cdot 12288(t(t-1))^{12}$, and if x, y are the ends of an M -path in $G - A$, then either $d(x, y) < 12t(t-1)$, or each of x, y lies at distance at most $6t(t-1) - 1$ from some vertex of Z . We define strips similarly as in the proof of Theorem 5.2, except that strips will now consist of $24t(t-1) + r$ consecutive paths. We construct walls $W_1, W_2, \dots, W_{t(t-1)}$, but this time each will be a $(24t(t-1) + r)$ -wall, and they will be pairwise at distance at least $12t(t-1)$. We construct the graphs H_i and cycles C_i as in the proof of Theorem 5.2, and apply the algorithm of Theorem 6.5 to each. If each of them has a C_i -cross, then we use those crosses to construct a model of K_t grasped by W , as in the proof of Theorem 5.2. On the other hand if some H_i has a C_i -reduction sequence leading to a C_i -reduction that can be drawn in the plane with C_i bounding a face, then we apply the algorithm of Lemma 6.6 to H_i , wall W_i , its outer cycle and the C_i -reduction sequence to produce a separation (X', Y) satisfying (1)–(4) of Lemma 5.1. Finally, we convert (X', Y) to a required separation of G as in the proof of Theorem 5.2. \square

For fixed t the most time-consuming step in the above algorithm is the use of Shiloach's algorithm stated as Theorem 6.5. Any improvement in the running time of that algorithm would immediately imply a corresponding improvement to our algorithm. Several faster algorithms for the task of Theorem 6.5 have been announced or appeared in conference proceedings and elsewhere, but without complete proofs. For instance, the algorithm of [17] uses a data structure whose analysis has never been published. We believe that there should be a linear-time algorithm, but none has been published thus far.

References

- [1] R. Diestel, Graph Decompositions - a Study in Infinite Graph Theory, Oxford University Press, Oxford (1990).
- [2] R. Diestel, Graph Theory, 3rd Edition, Springer, 2005.

- [3] A. Frank, Packing paths, cuts and circuits – a survey, in *Paths, Flows and VLSI-Layout*, B. Korte, L. Lovász, H. J. Promel and A. Schrijver (Eds.), Springer-Verlag, Berlin, 1990, 49–100.
- [4] H. A. Jung, Verallgemeinerung des n-Fachen Zusammenhangs für Graphen, *Math. Ann.* **187** (1970), 95–103.
- [5] R. M. Karp, On the computational complexity of combinatorial problems, *Networks* **5** (1975), 45–68.
- [6] K. Kawarabayashi, Y. Kobayashi and B. Reed, The disjoint paths problem in quadratic time, *submitted*.
- [7] J. F. Lynch, The equivalence of theorem proving and the interconnection problem, *ACM SIGDA Newsletter* **5** (1975), 31–65.
- [8] N. Robertson and P. D. Seymour, An outline of a disjoint paths algorithm, in *Paths, Flows, and VLSI-Layout*, B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver (Eds.), Springer-Verlag, Berlin, 1990, 267–292.
- [9] N. Robertson and P. D. Seymour, Graph minors. V. Excluding a planar graph, *J. Combin. Theory Ser. B*, **41** (1986), 92–114.
- [10] N. Robertson and P. D. Seymour, Graph minors IX. Disjoint crossed paths, *J. Combin. Theory Ser. B*, **49** (1990), 40–77.
- [11] N. Robertson and P. D. Seymour, Graph minors. XIII. The disjoint paths problem, *J. Combin. Theory Ser. B* **63** (1995), 65–110.
- [12] N. Robertson, P. D. Seymour, and R. Thomas, Quickly excluding a planar graph, *J. Combin. Theory Ser. B*, **62** (1994), 323–348.
- [13] N. Robertson and P. D. Seymour, Graph minors. XVI. Excluding a non-planar graph, *J. Combin. Theory Ser. B* **89** (2003), 43–76.
- [14] A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency, number 24 in Algorithm and Combinatorics, Springer Verlag, 2003.
- [15] P. D. Seymour, Disjoint paths in graphs, *Discrete Math.* **29** (1980), 293–309.
- [16] Y. Shiloach, A polynomial solution to the undirected two paths problem, *J. Assoc. Comp. Machinery*, **27**, (1980), 445–456.
- [17] T. Tholey, Solving the 2-disjoint paths problem in nearly linear time, *Theory Comput. System.* **39** (2006), 51–78.
- [18] C. Thomassen, 2-linked graphs, *Europ. J. Combinatorics* **1** (1980), 371–378.
- [19] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937), 570–590.
- [20] K. Wagner, Über eine Erweiterung des Satzes von Kuratowski, *Deutsche Math.* **2** (1937), 280–285.